Chapter review 7

1 a $\sum_{r=1}^{10} r = \frac{1}{2}\times 10\times 11 = 55$

b $\sum_{r=10}^{50} r = \sum_{r=1}^{50} r - \sum_{r=1}^{9} r$

$= \frac{1}{2}\times 50\times 51 - \frac{1}{2}\times 9\times 10$

$= 1230$

c $\sum_{r=1}^{10} r^2 = \frac{1}{6}\times 10\times 11\times 21 = 385$

d $\sum_{r=1}^{10} r^3 = \frac{1}{4}\times 10^2\times 11^2 = 3025$

e $\sum_{r=26}^{50} r^2 = \sum_{r=1}^{50} r^2 - \sum_{r=1}^{25} r^2$

$= \frac{1}{6}\times 50\times 51\times 101 - \frac{1}{6}\times 25\times 26\times 51$

$= 37\,400$

f $\sum_{r=50}^{100} r^3 = \sum_{r=1}^{100} r^3 - \sum_{r=1}^{49} r^3$

$= \frac{1}{4}\times 100^2\times 101^2 - \frac{1}{4}\times 49^2\times 50^2$

$= 24\,001\,875$

g $\sum_{r=1}^{60} r + \sum_{r=1}^{60} r^2$

$= \frac{1}{2}\times 60\times 61 + \frac{1}{6}\times 60\times 61\times 121$

$= 75\,640$

2 a $\sum_{r=1}^{n} (3r-5) = 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 5$

$= \frac{3}{2}n(n+1) - 5n$

$= \frac{3}{2}n^2 + \frac{3}{2}n - 5n$

$= \frac{3}{2}n^2 - \frac{7}{2}n$

2 b $\sum_{r=1}^{n}(r^2+r)=\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r$

$$=\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)$$
$$=\frac{1}{6}n(n+1)(2n+1+3)$$
$$=\frac{1}{6}n(n+1)(2n+4)$$
$$=\frac{1}{3}n(n+1)(n+2)$$

c $\sum_{r=1}^{n}(3r^2+7r)=3\sum_{r=1}^{n}r^2+7\sum_{r=1}^{n}r$

$$=\frac{3}{6}n(n+1)(2n+1)+\frac{7}{2}n(n+1)$$
$$=\frac{1}{6}n(n+1)(6n+3+21)$$
$$=\frac{1}{6}n(n+1)(6n+24)$$
$$=n(n+1)(n+4)$$

d $\sum_{r=1}^{n}(4r^3+6r^2)=4\sum_{r=1}^{n}r^3+6\sum_{r=1}^{n}r^2$

$$=n^2(n+1)^2+n(n+1)(2n+1)$$
$$=n(n+1)(n^2+n+2n+1)$$
$$=n(n+1)(n^2+3n+1)$$

e $\sum_{r=1}^{n}(r^2-2r)=\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}r$

$$=\frac{1}{6}n(n+1)(2n+1)-n(n+1)$$
$$=\frac{1}{6}n(n+1)(2n+1-6)$$
$$=\frac{1}{6}n(n+1)(2n-5)$$

2 f $$\sum_{r=1}^{n}\left(2r^2-3r\right)=2\sum_{r=1}^{n}r^2-3\sum_{r=1}^{n}r$$
$$=\frac{2}{6}n\left(n+1\right)\left(2n+1\right)-\frac{3}{2}n\left(n+1\right)$$
$$=\frac{n}{6}\left(n+1\right)\left(2\left(2n+1\right)-3(3)\right)$$
$$=\frac{n}{6}\left(n+1\right)\left(4n-7\right)$$

g $$\sum_{r=1}^{n}(r^2-5)=\sum_{r=1}^{n}r^2-5\sum_{r=1}^{n}1$$
$$=\frac{1}{6}n(n+1)(2n+1)-5n$$
$$=\frac{1}{6}n(2n^2+3n+1-30)$$
$$=\frac{1}{6}n(2n^2+3n-29)$$

h $$\sum_{r=1}^{n}(2r^3+3r^2+r+4)$$
$$=2\sum_{r=1}^{n}r^3+3\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}r+4\sum_{r=1}^{n}1$$
$$=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)(2n+1)$$
$$+\frac{1}{2}n(n+1)+4n$$
$$=\frac{1}{2}n(n^3+2n^2+n+n^2+3n+1+n+1+8)$$
$$=\frac{1}{2}n(n^3+3n^2+5n+10)$$

3 $\sum_{r=1}^{30} r(3r-1) = \sum_{r=1}^{30} 3r^2 - r$

$= 3\sum_{r=1}^{30} r^2 - \sum_{r=1}^{30} r$

$= \frac{1}{2}\times 30\times 31\times 61 - \frac{1}{2}\times 30\times 31$

$= 27\,900$

4 a $\sum_{r=1}^{n} r^2(r-3) = \sum_{r=1}^{n} r^3 - 3r^2$

$= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r^2$

$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)(2n+1)$

$= \frac{1}{4}n(n+1)(n(n+1) - 2(2n+1))$

$= \frac{1}{4}n(n+1)(n^2 - 3n - 2)$

so $a = -3,\ b = -2.$

b $\sum_{r=1}^{20} r^2(r-3) = \frac{1}{4}\times 20\times 21\times (20^2 - 60 - 2)$

$= 35490$

5 a $\sum_{r=1}^{n} (2r-1)^2 = \sum_{r=1}^{n} 4r^2 - 4r + 1$

$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$

$= \frac{1}{3}n(n+1)(2(2n+1) - 6) + n$

$= \frac{1}{3}n(n+1)(4n-4) + n$

$= \frac{1}{3}n(4n^2 - 4 + 3)$

$= \frac{1}{3}n(2n-1)(2n+1)$

b $\sum_{r=1}^{2n} (2r-1)^2 = \frac{1}{3}\times 2n(4n-1)(4n+1)$

$= \frac{2}{3}n(4n-1)(4n+1)$

6 a $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2r$

$= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{1}{6}n(n+1)(2n+1) + n(n+1)$

$= \frac{1}{6}n(n+1)(2n+7)$

b $\sum_{r=15}^{30} r(r+2) = \sum_{r=1}^{30} r(r+2) - \sum_{r=1}^{14} r(r+2)$

$= \frac{1}{6} \times 30 \times 31 \times 67 - \frac{1}{6} \times 14 \times 15 \times 35$

$= 9160$

7 a $\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$

$= \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{6}n(2n+1)(2(4n+1) - n - 1)$

$= \frac{1}{6}n(2n+1)(7n+1)$

b $\sum_{r=16}^{30} r^2 = \frac{1}{6} \times 15 \times 31 \times 106 = 8215$

8 a $\sum_{r=1}^{n} (r^2 - r - 1) = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$

$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n$

$= \frac{1}{3}n(n^2 - 4)$

b $\sum_{r=10}^{40} (r^2 - r - 1)$

$= \sum_{r=1}^{40} (r^2 - r - 1) - \sum_{r=1}^{9} (r^2 - r - 1)$

$= \frac{1}{3} \times 40 \times (40^2 - 4) - \frac{1}{3} \times 9 \times (9^2 - 4)$

$= 21\,049$

8 c $\sum_{r=1}^{n}(r^2-r-1)=\sum_{r=1}^{2n}r$

$\Rightarrow \frac{1}{3}n(n^2-4)=\frac{1}{2}\times 2n\times(2n+1)$

$\Rightarrow \frac{1}{3}n(n^2-4)=n(2n+1)$

$\Rightarrow \frac{1}{3}(n^2-4)=2n+1$

$\Rightarrow n^2-4=6n+3$

$\Rightarrow n^2-6n-7=0$

$\Rightarrow (n-7)(n+1)=0$

$\Rightarrow n=7$

9 a $\sum_{r=1}^{n}r(2r^2+1)=\sum_{r=1}^{n}2r^3+r$

$=2\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r$

$=\frac{1}{2}n^2(n+1)^2+\frac{1}{2}n(n+1)$

$=\frac{1}{2}n(n+1)(n(n+1)+1)$

$=\frac{1}{2}n(n+1)(n^2+n+1)$

b $\sum_{r=1}^{n}(100r^2-r)=100\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r$

$=\frac{1}{6}n(n+1)(200n+97)$

Now if $\sum_{r=1}^{n}r(2r^2+1)=\sum_{r=1}^{n}(100r^2-r)$, then

$\frac{1}{2}n(n+1)(n^2+n+1)=\frac{1}{6}n(n+1)(200n+97)$

$\frac{1}{6}n(n+1)(3(n^2+n+1)-(200n+97))=0$

$\frac{1}{6}n(n+1)(3n^2-197n-94)=0$

But $n\neq 0$, $n\neq -1$ and $3n^2-197n-94$ has discriminant 39 937 which is not square.

10 a $\sum_{r=1}^{n} r(r+1)^2 = \sum_{r=1}^{n} r^3 + 2r^2 + r$

$= \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$

$= \frac{1}{12}n(n+1)(n+2)(3n+5)$

b $\sum_{r=1}^{n} 70r = 35n(n+1)$

$\Rightarrow \frac{1}{12}n(n+1)(n+2)(3n+5) = 35n(n+1)$

$\Rightarrow \frac{1}{12}(n+2)(3n+5) = 35$

$\Rightarrow 3n^2 + 11n + 10 = 420$

$\Rightarrow 3n^2 + 11n - 410 = 0$

$\Rightarrow (3n+41)(n-10) = 0$

$\Rightarrow n = 10$

11 $\frac{1}{6}n(n+1)(2n+1) = \frac{9}{2}(n+1)(n+2) + n + 1$

$\Rightarrow n(n+1)(2n+1) = 27(n+1)(n+2) + 6(n+1)$

$\Rightarrow (n+1)(2n^2 + n) = (n+1)(27n + 54 + 6)$

$\Rightarrow 2n^2 + n = 27n + 60$

$\Rightarrow 2n^2 - 26n - 60 = 0$

$\Rightarrow n^2 - 13n - 30 = 0$

$\Rightarrow (n-15)(n+2) = 0$

$\Rightarrow n = 15$

Challenge

a $\sum_{i=1}^{n}\left(\sum_{r=1}^{i} r^2\right) = \sum_{i=1}^{n} \frac{1}{6} i(i+1)(2i+1)$

$= \frac{1}{6}\sum_{i=1}^{n} 2i^3 + 3i^2 + i$

$= \frac{1}{6}\left(\frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\right)$

$= \frac{1}{12}\left(n(n+1)(n^2 + n + 2n + 1 + 1)\right)$

$= \frac{1}{12}n(n+1)(n^2 + 3n + 2)$

$= \frac{1}{12}n(n+1)(n+2)(n+1)$

$= \frac{1}{12}n(n+1)^2(n+2)$

b $\sum_{j=1}^{n}\left(\sum_{i=1}^{j}\left(\sum_{r=1}^{i} r\right)\right) = \sum_{j=1}^{n}\left(\frac{1}{2}\sum_{i=1}^{j} i(i+1)\right)$

$= \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} i^2\right) + \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} i\right)$

$= \frac{1}{24}n(n+1)^2(n+2) + \frac{1}{4}\sum_{j=1}^{n} j^2 + \frac{1}{4}\sum_{j=1}^{n} j$

$= \frac{1}{24}n(n+1)^2(n+2) + \frac{1}{24}n(n+1)(2n+1)$

$+ \frac{1}{8}n(n+1)$

$= \frac{1}{24}n(n+1)(n^2 + 5n + 6)$

$= \frac{1}{24}n(n+1)(n+2)(n+3)$